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On symmetric unions and alternating knots

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Abstract. In this paper, we introduce the partial alternating number for a symmetric union and show that for any positive integer n, there exists a symmetric union with partial alternating number $\geq n$. We also show that there exist infinitely many symmetric unions with partial alternating number one.

1 Introduction

The union of knots was introduced by S. Kinoshita and H. Terasaka [2]. A *symmetric union* [3] is a knot which is obtained from the connected sum of a knot and its mirror image by inserting some vertical twists along the symmetry axis to the diagram. A symmetric union is known to be a *ribbon knot* [4]. In this paper, we study symmetric unions for alternating knots. For any symmetric union, we have only finitely many alternating partial knots for the symmetric union presentations (Proposition 3.1). We define the *partial alternating number* for a symmetric union as the number of alternating knots (up to mirror images). Then we have the following.

Theorem 1.1. *For any positive interger n, there exists a symmetric union with partial alternating number $\geq n$.*

The notation for prime knots up to 10 crossings is due to Rolfsen's book [1]. We denote the mirror image of a knot K by K^*. In Section 2, we shall define a symmetric union. In Section 3, we shall define the partial alternating number for a symmetric union and prove Theorem 1.1. In Section 4, we shall show that there exist infinitely many symmetric unions with partial alternating number one.

2 Definition

We define a symmetirc union [3] as follows. We denote the tangles made of half twists by integers $n \in \mathbb{Z}$ and the horizontal trivial tangle by ∞ as in Figure 1.

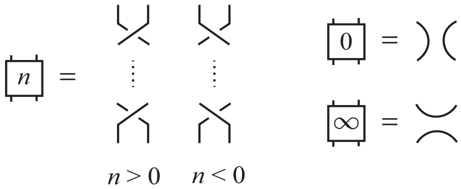

Figure 1: Tangles.

Definition 2.1. Let D_K be an unoriented diagram of knot K and D_K^*, the diagram D_K reflected at an axis in the plane. We take k 0-tangles T_i $(i = 0, \ldots, k)$ on the symmetry axis as in Figure 2(a). Then we replace the tangles T_i with $T_0 = \infty$ and $T_i = n_i \in \mathbb{Z}$ for $i = 1, \ldots, k$ as in Figure 2(b). We call the resultant diagram a *symmetric union* and write $D_K \cup D_K^*$ $(n_1, \ldots, n_k)$ and the diagram is called a *symmetric union presentation.* The knot K is called the *partial knot* for the symmetric union presentation. We say that a knot K is a *symmetric union* if K has a symmetric union presentation.

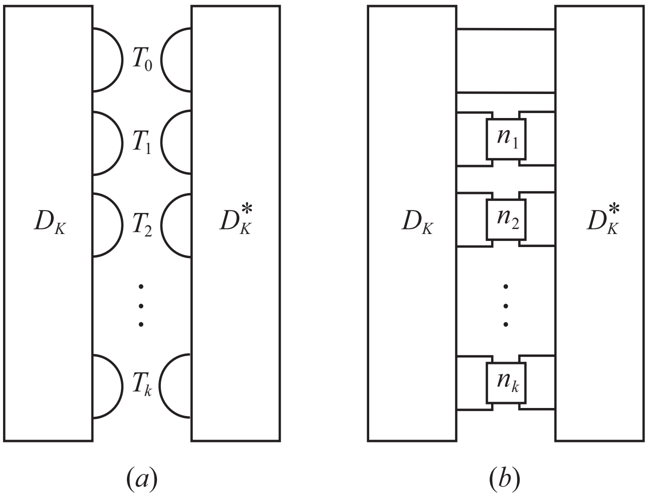

Figure 2: A symmetric union.

3 Symmetric unions and alternating knots

Proposition 3.1. *For any symmetric union, we have only finitely many alternating partial knots for the symmetric union presentations.*

Proof. Let $\overline{K}$ be a knot with a symmetric union presentation with K as a partial knot. Then by [3, Theorem 2.6], we have $\det(\overline{K}) = \det(K)^2$, where $\det(K)$ is the *determinant* of K. Suppose that K is a (non-trivial) alternating knot. Then by [6] and [5, Chapter 8], we have $\deg Q_K(x) = c(K) - 1$, where $\deg Q_K$ is the maximal degree of the *Q-polynomial* of K. Since any alternating knot is quasi-alternating, by [7], we have $\deg Q_K(x) < \det(K)$. Thus we obtain that $c(K) - 1 \leq (c(K) - 1)^2 < (\det(K))^2 = \det(\overline{K})$.

Definition 3.2. Let K be a symmetric union. Then the number of alternating partial knots (excluding mirror images) for the symmetric union presentations for K is called the *partial alternating number* for K.

Remark 3.3. Clearly, the partial alternating number is an invariant of a symmetric union. For example, the connected sum of a prime alternating knot and its mirror image is a symmetric union with partial alternating number ≥ 1. In particular, the partial alternating number of the connected sum of 3_1 and its mirror image is equal to one. However we do not know if the equality holds in general.

Proof of Theorem 1.1. For a positive integer $m \geq 2$, let $\overline{K}_m$ be a knot which has the symmetric union presentation $\overline{D}_m$ as shown in Figure 3.

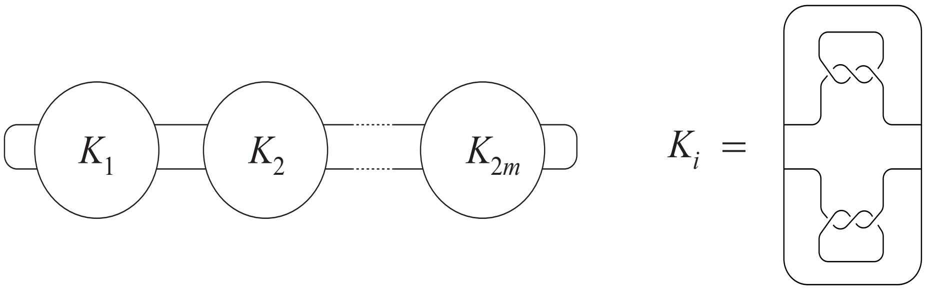

Figure 3: $\overline{K}_m$.

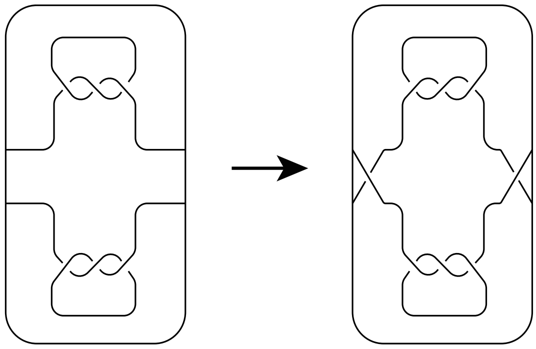

Figure 4: A flype.

Note that $\overline{D}_m$ has $\sharp_{i=1}^{2m} 3_1$, which is the connected sum of $2m$ $3_1's$, as the partial knot. Then for an integer p $(1 \leq p \leq m)$, by flyping $K_1, \ldots, K_p$ to $\overline{D}_m$ as shown in Figure 4, we have a symmetric union presentation with the partial knot $(\sharp_{i=1}^{p} 3_1^*)\sharp(\sharp_{i=1}^{2m-p} 3_1)$ denoted by H_p.

Then it is easily seen that the *Jones polynomials* [5] of H_{p_1} and H_{p_2} are distinct if $p_1 \neq p_2$. So we have at least m distinct alternating partial knots (up to mirror images) for $\overline{K}_m$.

4 Examples

Proposition 4.1. *There exist infinitely many symmetric unions with partial alternating number one.*

Proof. For a positive integer n, let $\hat{K}_n$ be a knot which has the symmetric union presentation $D \cup D^*(n)$ as shown in Figure 5. Suppose that K is a partial knot for a symmetric union presentation for $\hat{K}_n$. By using the same method as in the proof of Theorem 1.1, we know that $c(K) - 1 < \det(K) = 3$. Since $c(K) \leq 3$ and $\det(K) = 3$, K is either 3_1 or 3_1^*. Thus we know that the partial alternating number of $\hat{K}_n$ is equal to one. By using the Jones polynomial, we know that $\hat{K}_{n_1} \neq \hat{K}_{n_2}$ if $n_1 \neq n_2$.

Example 4.2. Let $\overline{K}$ be the connected sum of 4_1 and 4_1^* and let K be the partial knot for a symmetric union presentation for $\overline{K}$. Then as in the proof of Proposition 4.1, we know that $c(K) - 1 < \det(K) = 5$. There are only four alternating knots 4_1, 5_1, 4_1^* and 5_1^* such that the crossing numbers ≤ 5 and the determinants are equal to 5. Thus we know that the partial alternating number of $\overline{K}$ is ≤ 2.

Let $\tilde{K}_n$ be a knot which has the symmetric union presentation $D \cup D^*(n)$ as shown in Figure 6. Suppose that K is a partial knot for a symmetric union presentation for $\tilde{K}_n$. As in the proof of Proposition 4.1, we know that $c(K) - 1 < \det(K) = 9$. There are only four alternating knots, $6_1, 9_1, 3_1\sharp 3_1$ and $3_1\sharp 3_1^*$, such that the crossing numbers ≤ 9 and the determinants are equal to 9, up to mirror images. Thus we know that the partial alternating number of $\tilde{K}_n$ is ≤ 4. By using the same method as in the proof of Theorem 1.1, we know that $3_1\sharp 3_1$ and $3_1\sharp 3_1^*$ are the partial knots for $\tilde{K}_n$. So we know that the partial alternating number of $\tilde{K}_n$ is ≥ 2.

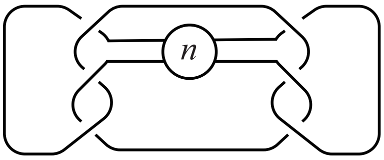

Figure 5: $\hat{K}_n$.

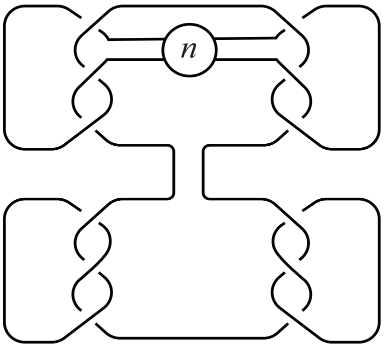

Figure 6: $\tilde{K}_n$.

Question. Can the knots 6_1 and 9_1 be partial knots for $\tilde{K}_n$?

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