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On alternating partial knots of symmetric unions

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Abstract. In this paper, we consider the set of alternating partial knots for a symmetric union. We study the sets for all prime symmetric unions up to 10 crossings.

1 Introduction

The *union* of knots was introduced by S. Kinoshita and H. Terasaka [2]. A *symmetric union* [4] is a knot which is obtained from the connected sum of a knot and its mirror image by inserting some vertical twists along the symmetry axis to the diagram. A symmetric union is known to be a *ribbon knot* [4]. In this paper, we study alternating knots for symmetric unions. For any symmetric union, we have only finitely many alternating partial knots for the symmetric union presentations (Proposition 3.1). We denote the set of partial alternating knots (up to reversing orientation or taking mirror image) for a symmetric union K by $\Gamma_a(K)$. Then we have the following.

Theorem 1.1.

(1) $\Gamma_a(8_{20}) = \Gamma_a(9_{46}) = \Gamma_a(10_{140}) = \{3_1\}$,

(2) $\Gamma_a(9_{41}),\ \Gamma_a(10_{48}) \subset \{5_2, 7_1\}$,

(3) $\Gamma_a(10_3),\ \Gamma_a(10_{129}),\ \Gamma_a(10_{137}),\ \Gamma_a(10_{155}) \subset \{4_1, 5_1\}$,

(4) $\Gamma_a(10_{75}),\ \Gamma_a(10_{87}),\ \Gamma_a(10_{99}) \subset \{6_1, 9_1, 3_1\sharp 3_1, 3_1\sharp - 3_1\}$,

(5) $\Gamma_a(10_{123}) \subset \{6_2, 7_2, 11a_{367}\}$,

(6) $\Gamma_a(0_1) = \Gamma_a(10_{153}) = \{0_1\}$.

The notation for prime knots up to 10 crossings is due to Rolfsen's book [6]. We denote the mirror image of a knot K (with reversed orientation) by $-K$. In Section 2, we shall define a symmetric union. In Section 3, we shall consider the set of alternating partial knots for a symmetric union and prove Theorem 3.3. In Section 4, we shall consider the partial knots for 10_{129}.

2 Definition

We define a symmetirc union [4] as follows. We denote the tangles made of half twists by integers $n \in \mathbb{Z}$ and the horizontal trivial tangle by ∞ as in Figure 1.

Definition 2.1. Let D_K be an unoriented diagram of a knot K and $-D_K$, the diagram D_K reflected at an axis in the plane. We take k 0-tangles T_i $(i = 0, \ldots, k)$ on the symmetry axis as in Figure 2(a). Then we replace the tangles T_i with $T_0 = \infty$ and $T_i = n_i \in \mathbb{Z}$ for $i = 1, \ldots, k$ as in Figure 2(b). We call the resultant diagram a *symmetric union* and write $D_K \cup -D_K$ $(n_1, \ldots, n_k)$ and the diagram is called a *symmetric union presentation*. The knot K is called the *partial knot* for the symmetric union presentation. We say that a knot K is a *symmetric union* if K has a symmetric union presentation.

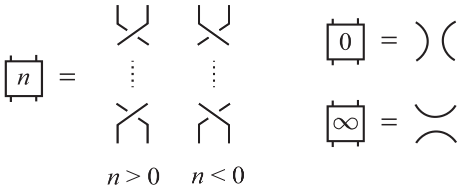

Figure 1: Tangles.

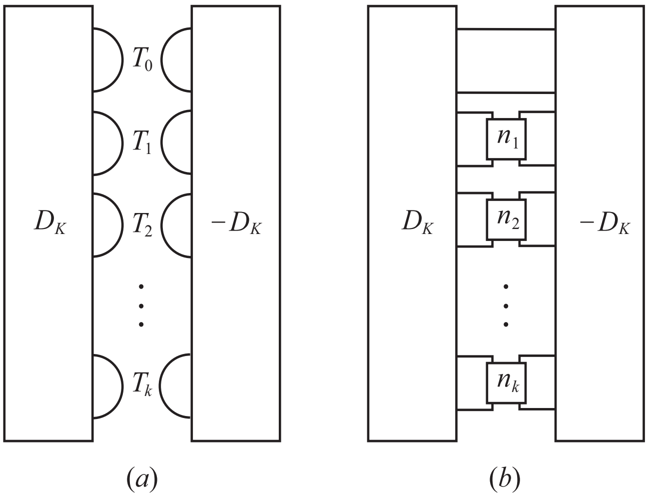

Figure 2: A symmetric union.

3 Symmetric unions and alternating knots

Proposition 3.1. [8] *For any symmetric union, we have only finitely many alternating partial knots for the symmetric union presentations.*

Let $\Gamma_a(K)$ be the set of alternating partial knots for a symmetric union K, up to reversing orientation or taking mirror image. By Proposition 3.1, we know that $\Gamma_a(K)$ is a finite set. We have 21 symmetric unions in the set of prime knots up to 10 crossings [4]
$\{6_1, 8_8, 8_9, 8_{20}, 9_{27}, 9_{41}, 9_{46}, 10_3, 10_{22}, 10_{35}, 10_{42}, 10_{48}, 10_{75}, 10_{87}, 10_{99}, 10_{123}, 10_{129}, 10_{137}, 10_{140}, 10_{153}, 10_{155}\}$.

Proof of Theorem 1.1. Since det(8_{20}) =det(9_{46}) =det(10_{140}) $= 9$, by [4, Theorem 2.6], the determinants of partial knots of them are 3. Then, we know that the minimal crossing numbers of alternating partial knots of 8_{20}, 9_{46} and 10_{140} are less than or equal to 3 since the determinant of an alternating knot is never smaller than its minimal crossing number ([1, Proposition 13.30]). Thus we know that the alternating partial knots of 8_{20}, 9_{46} and 10_{140} can be only 3_1 (up to reversing orientation or taking mirror image). In fact, 8_{20}, 9_{46} and 10_{140} have 3_1 as an alternating partial knot [4]. Thus we have $\Gamma_a(8_{20}) = \Gamma_a(9_{46}) = \Gamma_a(10_{140}) = \{3_1\}$.

Since det(9_{41}) =det(10_{48}) $= 49$, the determinants of partial knots of them are 7. Then, we know that the minimal crossing numbers of alternating partial knots of 9_{41} and 10_{48} are less than or equal to 7. Thus we know that the alternating partial knots of 9_{41} and 10_{48} can be 5_2 and 7_1 (up to reversing orientation or taking mirror image).

Since det(10_3) =det(10_{129}) =det(10_{137}) =det(10_{155}) $= 25$, the determinants of partial knots of them are 5.

Then, we know that the minimal crossing numbers of alternating partial knots of 10_3, 10_{129}, 10_{137} and 10_{155} are less than or equal to 5. Thus we know that the alternating partial knots of 9_{41} and 10_{48} can be 4_1 and 5_1 (up to reversing orientation or taking mirror image).

Since $\det(10_{75}) = \det(10_{87}) = \det(10_{99}) = 81$, the determinants of partial knots of them are 9. Then, we know that the minimal crossing numbers of alternating partial knots of 10_{75}, 10_{87} and 10_{99} are less than or equal to 9. Thus we know that the alternating partial knots of 10_{75}, 10_{87} and 10_{99} can be 6_1, 9_1, $3_1\sharp 3_1$ and $3_1\sharp -3_1$ (up to reversing orientation or taking mirror image).

Since $\det(10_{123}) = 121$, the determinants of partial knots of them are 11. Then, we know that the minimal crossing number of an alternating partial knot of 10_{123} is less than or equal to 11. Thus we know that the alternating partial knots of 10_{123} can be 6_2, 7_2 and $11a_{367}$ (up to reversing orientation or taking mirror image).

Since $\det(0_1) = \det(10_{153}) = 1$, the determinants of partial knots of them are 1. Then, we know that the minimal crossing numbers of alternating partial knots of 0_1 and 10_{153} are less than or equal to 1. Thus we know that the alternating partial knots of 0_1 and 10_{153} can only be 0_1. In fact, 0_1 and 10_{153} have 0_1 as an alternating partial knot [4].

Remark 3.2. By a result of [4], we know that $\Gamma_a(9_{41}),\ \Gamma_a(10_{48}) \ni 5_2$, $\Gamma_a(10_3) \ni 5_1$ and $\Gamma_a(10_{129}),\ \Gamma_a(10_{137}),\ \Gamma_a(10_{155}) \ni 4_1$. We can also have the following.

(1) $\Gamma_a(6_1) = \{3_1\}$,

(2) $\Gamma_a(8_8) = \Gamma_a(8_9) = \{4_1, 5_1\}$,

(3) $\Gamma_a(9_{27}) = \Gamma_a(10_{22}) = \Gamma_a(10_{35}) = \{5_1, 7_1\}$,

(4) $\Gamma_a(10_{42}) = \{6_1, 9_1\}$.

(We have obtained more general result in [5].)

Example 3.3. As shown in [8], there exists an infinite family $\{K_i\}$ of symmetric unions with $\Gamma_a(K_i) = \{3_1\}$. We consider the knot K_m described in Figure 3.

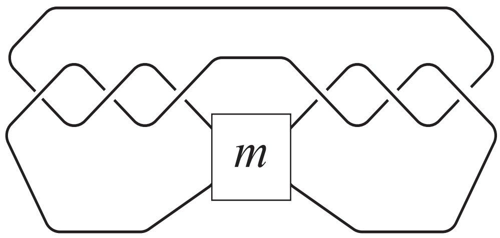

Figure 3: K_m

By [7, Theorem 1.1], we know that the Jones polynomial $V_{K_m}(t)$ of K_m is equal to

$$(-1)^m t^{-m}(t + t^3 - t^4)(t^{-1} + t^{-3} - t^{-4}) + (1 - (-1)^m t^{-m}).$$

Then the determinant of K_m is $|V_{K_m}(-1)| = 9$. By using the same method as in the proof of Proposition 3.3, we know that $\Gamma_a(K_m) = \{3_1\}$. It is easily seen that the maximal degree of $V_{K_m}(t)$ is equal to $3 - m$ if $m < 0$. Thus we know that K_{m_1} is not equivalent to K_{m_2} if $m_1, m_2 < 0$ and $m_1 \neq m_2$.

Remark 3.4. It is easily seen that K_m is a prezel knot $P(-3, m, 3)$ [3]. By [3, Theorem 2.3.1], we know that K_m is not a 2-bridge knot if $|m| \geq 2$.

4 Knots with symmetric union presentations with one twist region

We need the following results.

Theorem 4.1. [7] *Let $\overline{K}$ be a knot with a symmetric union presentation of the form $D_K \cup -D_K(m)$. Then* $t^{-m}V_{\overline{K}}(t) + (-1)^m V_{\overline{K}}(t^{-1}) = (t^m + (-1)^m)V_K(t)V_K(t^{-1})$.

Corollary 4.2. [7] *Let $\overline{K}$ be a knot with a symmetric union presentation of the form $D_K \cup -D_K(m)$. Then* $V'_{\overline{K}}(-1) \equiv 0 \bmod 8|m|$.

By a simple calculation, we have
$V_{10_{129}}(t) = -t^{-3} + 2t^{-2} - 3t^{-1} + 5 - 4t + 4t^2 - 3t^3 + 2t^4 - t^5$,
$V_{4_1}(t) = t^{-2} - t^{-1} + 1 - t + t^2$,
$V_{5_1}(t) = t^2 + t^4 - t^5 + t^6 - t^7$.
Suppose that 10_{129} has a symmetric union presentation of the form $D_K \cup -D_K(m)$. Then by Corollary 4.2, we know that $|m| = 1$ or 3 since $V'_{10_{129}}(-1) = -24$ and 10_{129} is a prime knot. So we may have the formula in Theorem 4.1 in the case when $|m| = 1$ or 3.

By Theorem 1.1, 10_{129} may have 4_1 and 5_1 as partial knots. In fact, in the case when $m = 1$, we have
$t^{-1}V_{10_{129}}(t) - V_{10_{129}}(t^{-1}) = (t-1)V_{5_1}(t)V_{5_1}(t^{-1})$.
In the case when $m = -1$, we have
$tV_{10_{129}}(t) - V_{10_{129}}(t^{-1}) = (t^{-1}-1)V_{4_1}(t)V_{4_1}(t^{-1})$.

Question. Does 10_{129} have a symmetric union presentation of the form $D_{5_1} \cup -D_{5_1}(1)$?

Remark 4.3. It is known that 10_{129} has 4_1 as an alternating partial knot [4]. If the answer to the above question is affirmative, then we have $\Gamma_a(10_{129}) = \{4_1, 5_1\}$.

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