On C−R submanifolds of $\mathbf{C}^n$ of C−R codimension 1.

Kiyoshi Shiga

Dept. of Math., Fac. of Gen. Educ., Gifu Univ.

(Received Oct., 31, 1974)

§ 0. Introduction

For a real hypersurface of a complex manifold, there exists a system of partial differential equations, called the tangential Cauchy-Riemann equations. The trace of a holomorphic function on the complex manifold satisfies the tangential Cauchy-Riemann equations, as easily follows from the definition (c. f. Definition 2-1). Some properties of solutions of the tangential Cauchy Riemann equations are known, concerning relations with holomorphic functions. We recall some fundamental results in §2.

We consider a class of real submanifolds of a complex manifolds, called C−R submanifolds, (c.f. Definition 1-1). For a real analytic C−R submanifold of C^n, there exists a complex submanifold, not necessarily closed, such that it contains the C−R submanifold generically (Tomassini [10]).

More generally we define a C−R structure on a real manifold by a complex distribution which satisfies some integrability condition (c.f. Definition 1-2). H. Rossi proved that a real analytic C−R manifold can be realised as a real submanifold of a complex manifold (H. Rossi [7]). He also proved that a compact connected strongly pseudoconvex C−R manifold of C−R codimension 1 can be realized as a boundary of a strongly pseudoconvex domain of a Stein space with at most finite singularities, if its C−R dimension is greater than 2.

In this article we consider differentiable[1] C−R submanifolds of $\mathbf{C}^n$, and we obtain some generalization of H. Rossi's theorem and Tomassini's theorem in the case of C−R codimension 1, using results concerning the tangential Cauchy Riemann equations and H. Rossi's Stein completion.

§ 1. C−R structure

In this section we recall the definition of C−R structures. Let N be a complex manifold of dimension n, and M a real submanifold of N of dimension m. J denotes

1) In this paper, "differentiable" means C^∞-differentiable.

the almost complex structure of N and $T(N)$, $T(M)$ denotes the tangent bundle of N, M, respectively. At each point x of N, J induces a linear automorphism of $T_x(N)$ and $T_x(N)$ has a structure of a complex vector space induced by J. C_x denotes the maximal complex subspace containd in $T_x(M)$ for a point x of M. i.e.

$$C_x = T_x(M) \cap JT_x(M).$$

$T^c_x(N)$, $T^c_x(M)$ denotes the complexified tangent space of N, M at x, respectively. Put $H_x = \{ X - \sqrt{-1}JX \mid X \varepsilon C_x \}$, $\overline{H}_x = \{ X + \sqrt{-1}JX \mid X \varepsilon C_x \}$ and we call them the holomorphic and antiholomorphic tangent space of M at x, respectively.

Definition 1−1.

We will say M be a *C−R submanifold* of N, if $\dim._C H_x = \frac{1}{2}\dim._R C_x$ is constant on M. In this case $H(M) = \bigcup_{x\varepsilon M} H_x$, $\overline{H}(M) = \bigcup_{x\varepsilon M} \overline{H}_x$ are subbundles of $T^c(M)$. We will say l=rank $H(M)$, the C−R dimension of M, and r=dim.M−2rank $H(M) = m - 2\ell$, the C−R codimension of M.

We denote $\mathscr{H}(M)$, $\overline{\mathscr{H}}(M)$, the sheaf of germs of differentiable cross-sections of $H(M)$, $\overline{H}(M)$, respectively. Using the integrability of the almost complex structure J, we can easily show

$$[\mathscr{H}(M)_x \ , \mathscr{H}(M)_x] \subset \mathscr{H}(M)_x$$
$$[\overline{\mathscr{H}}(M)_x \ , \overline{\mathscr{H}}(M)_x] \subset \overline{\mathscr{H}}(M)_x.$$

We note that a real hypcrsureface of N is a C−R submanifold of N of C−R codimension 1.

More generally we define C−R manifolds. Let M be a real manifold. We consider a complex distribution $H(M) \subset T^c(M)$ on M.

Definition 1−2.

We will say that $(M, H(M))$ defines a *C−R manifold* of dimension m C−R dimension ℓ, if the following conditions are satisfied.

1. dim.$M = m$, rank $H(M) = l$.
2. $H(M)_x \cap \overline{H}(M)_x = 0$ for every $x \varepsilon M$, where $\overline{H}(M)_x$ is the conjugate subspace of $H(M)_x$ with respect to $T_x(M)$.
3. $[\mathscr{H}(M)_x \ , \mathscr{H}(M)_x] \subset \mathscr{H}(M)_x$ for every $x \varepsilon M$, where $\mathscr{H}(M)$ denote the sheaf of germs of differentiable cross-sections of $H(M)$ as before.

We will say $r = m - 2l$, C−R codimension of $(M, H(M))$.

Let M, N be differentiable manifolds, and $f: M \longrightarrow N$ a differentiable mapping M into N. Then f induces the differential of f, $f_* : T^c(M) \longrightarrow T^c(N)$.

Definition 1−3.

Let $(M, H(M))$, $(N, H(N))$ be C−R manifolds, and f a differentiable mapping M into N. We will say f is a *C−R mapping* if $f_*(H(M)) \subset H(N)$. If f is a diffeomorphism and f and f^{-1} are C−R mappings, we will say f is a C−R *diffeomorphism*.

A C−R submanifold of a complex manifold is a C−R manifold. But the converse is not true in general. In the real analytic case, the converse is also true (*H.*

Rossi [7]).

In the following, we only consider the case of C−R codimension 1. We will say these C−R manifolds, *C−R hypersurfaces.*

§ 2. Levi convexity and the tangential Cauchy-Rieemann equations.

Let $(M, H(M))$ be a C−R hypersurface of dimension n. Let π_x be the projection from $T_x^c(M)$ to $T_x^c(M)/H(M)_x \oplus \overline{H}(M)_x$. We define a hermitian form on $H(M)_x$ by

$$h_x(X_x, Y_x) = \sqrt{-1}\pi_x[X, \overline{Y}]_x \qquad X_x, Y_x \in H(M)_x$$

where X, Y are local differentiable cross-sections of $H(M)$ such that $X(x) = X_x$, $Y(x) = Y_x$. Since $\dim. T_x^c(M)/H(M)_x \oplus \overline{H}(M)_x = 1$, we identify $T_x^c(M)/H(M)_x \oplus \overline{H}(M)_x$ and $\mathbf{C}$, taking a real bace of $T_x^c(M)/H(M)_x \oplus \overline{H}(M)_x$. We call the above hermitian form, *the Levi form* of $(M, H(M))$ at x. We call $(M, H(M))$ is pseudoconvex, strongly pseudoconvex at x, if the Levi form is semi-definite, definite, respectively. We call $(M, H(M))$ is pseudoconvex, strongly pseudoconvex, if $(M, H(M))$ is pseudoconvex, strongly pseudoconvex at each point of M.

In the case of real hypersurfaces of complex manifolds, we can define the Levi form by another way. Let M be a real hypersurface of $D \subset \mathbf{C}^n$. There exists a real valued function φ such that

$$M = \{x \in D \mid \varphi(x) = 0\} \quad \text{and}$$
$$d\varphi \neq 0 \quad \text{on } M.$$

Let $(z_1, \cdots, z_n)$ be a coordinate system of $\mathbf{C}^n$, we consider the following hermitian form on $H(M)_x$.

$$\mathrm{L}(X) = \sum_{i,j} \frac{\partial^2 \varphi}{\partial z_i \partial \overline{z}_j} \xi_i \overline{\xi}_j$$

where $X = \sum_i \xi_i \frac{\partial}{\partial z_i} \in H(M)_x$ i.e. $\sum_i \frac{\partial \varphi}{\partial z_i} \xi_i = 0$.

The properties of semi-definitness, definiteness of the above hermitian form are independent of a choice of φ and a coordinate system of D. We call the above hermitian form, *the Levi form* of M at x. Two definitions of Levi form. are essentially the same in this case (c.f. Sommer [8]).

Let $(M, H(M))$ be a C−R hypersurface.

Definition 2−1.

We call a differentiable function, say f, satisfies *the tangential Cauchy-Riemann equations at* x if $Xf = 0$ for any element X of $\overline{H}(M)_x$. And we will say f, a *C−R function* if f satisfies the tangential Cauchy-Riemann equations at each point of M.

We recall some fundamental results concerning the tangential Cauchy-Riemann equations.

Let M be a pseudoconvex real hypersurface of a domain $D \subset \mathbf{C}^n$ $(n > 1)$, and we take a real valued differentiable function φ on D such that $M = \{x \in D \mid \varphi(x) = 0\}$ and $d\varphi \neq 0$ on M and the Levi form defined by φ is positive semi-definite.

Put $D^- = \{x \in D \mid \varphi(x) < 0\}$ and $D^+ = \{x \in D \mid \varphi(x) > 0\}$.

Theorem 2−2. (Wells [11], Hörmander [5], Andreotti-Hill [1])

Assume the Levi form of M has at least one positive eigen value at each point of M. Then there exists a domain $U \subset D^-$ *such that* $M \subset \partial U$ *and U has the following property. If f is a C−R function on M, then there exists a holomorphic function* $\tilde{f}$ *such that* $\tilde{f}$ *is differentiable up to M and* $\tilde{f} \mid M = f$.

We can easily show that the above holomorphic extention f of f is unique.

Next we consider C−R mappings, Let M, N be real hypersurfaces of D, $E \subset \mathbf{C}^n$ respectively. We assume M, N are pseudoconvex, and choose real valued functions φ, ψ and define $D^{\pm}$, $E^{\pm}$ as above.

Theorem 2−3.

Assume M, N are pseudo-convex and the Levi forms of M, N have at least one positive eigen value at each point of M, N. Let $F: M \longrightarrow N$ *be a C−R diffeomorphism. Then there exist domains* $U \subset D^-$, $V \subset E^-$ *and biholomorphic mapping* $\tilde{F}: U \longrightarrow V$ *such that* $\tilde{F}$ *is differentiable up to M and* $\tilde{F} \mid M = F$.

Proof. Let $(z_1, \cdots, z_n)$, $(w_1, \cdots, w_n)$ be a coordinate system of D, E, respectively. Then w_i $(1 \leqq i \leqq n)$ is a C−R function on N. Since F is a C−R diffeomorphism, $F^* w_i$ is a C−R function on M. We choose a domain $U' \subset D^-$ such that the property of theorem 2-2 holds, Let w_i be the holomorphic extension of $F^* w_i$. We define a holomorphic mapping $F: U' \longrightarrow \mathbf{C}^n$ by

$$F(p) = (w_1(p), \cdots, w_n(p));\ p \in U'.$$

Clealy $\tilde{F}$ is the holomorphic extension of F. Since F is a C−R diffeomorphism, and $\tilde{F}$ is a holomorphic mapping, we can show $\tilde{F}$ is maximal rank on a neighbourhood U' of M in $M \cup U'$, by the same method in [9]. For $p \in M$ there exists a neighbourhood U_p of p such that $U_p \cap D^- \subset U'$, and $\tilde{F} \mid U_p \cap D^-$ is a diffeomorphism. Since $\tilde{F}(M) = N$, $\tilde{F}(U_p \cap D^-) \subset E^-$ or $\tilde{F}(U_p \cap D^-) \subset E^+$. We shall show $F(U_p \cap D^-) \subset E^-$. We consider $F^* \psi$. $F^* \psi$ is differentiable on $U_p \cap \overline{D^-}$, so we can extend it differentiably on U_p, and we denote it $\tilde{\psi}$. We may assume $M \cap U_p = \{x \in U_p \mid \tilde{\psi}(x) = 0\}$. Put $U_p^{\pm} = \{x \in U_p \mid \tilde{\psi}(x) \gtrless 0\}$. Since $d\tilde{\psi} \neq 0$ on $M \cap U_p$, because of the invariance of the Levi form we conclude $U_p^- \subset D^-$. This means $\tilde{F}(U_p \cap D^-) \subset E^-$.

Then replacing U by a smaller neighbourhood if necessary, we may conclude $\tilde{F}(U) \subset E^-$, and $\tilde{F} \mid U$ is a diffeomorphism. Put $\tilde{F}(U) = V$. Then $\tilde{F}: U \longrightarrow V$ has the disiered property. q.e.d.

§ 3. C−R submanifolds of $\mathbf{C}^n$

In this section we prove our main theorem. Let M be a compact C−R submanifold of $\mathbf{C}^n$ of C−R dimension m $(m \geqq 1)$ and C−R codimension 1. $(M, H(M))$ be the C−R structure induced from the complex structure of $\mathbf{C}^n$. For a point $p \in M$, C_p is a $2m$ dimensional subspace of $T_p(M)$ and $\dim. T_p(M) = 2m+1$, then we can take a global coordinate system $(z_1, \cdots, z_n)$ of $\mathbf{C}^n$ such that $(\frac{\partial}{\partial z_1})_p$, $\cdots$, $(\frac{\partial}{\partial z_m})_p$ span $H(M)_p$ and $(\frac{\partial}{\partial x_1})_p$, $\cdots$, $(\frac{\partial}{\partial x_m})_p$, $(\frac{\partial}{\partial y_1})_p$, $\cdots$, $(\frac{\partial}{\partial y_{m+1}})_p$

span $T_p(M)$, where $z_i = x_i + y_i$. First we show that we can locally realize $(M,H(M))$ as a real hypersurface of $\mathbf{C}^{m+1}$.

Lemma 3 − 1.

For any point $p \in M$, there exists a neighbourhood U of p such that U is C−R diffeomorphic to a real hypersurface of a domain of $\mathbf{C}^{m+1}$.

Proof, We take a global coordinate system of $\mathbf{C}^n$ as above. z_i $(1 \leqq i \leqq m+1)$, as a function on M, is a C−R function. We define a mapping f from M into $\mathbf{C}^{m+1}$ defined by

$$f(q) = (z_1(q), \cdots, z_{m+1}(q)),\ q \in M.$$

Since $z_i(1 \leqq i \leqq m+1)$ is a C−R function, f is a C−R mapping. $f_* T_p(M)$ is spanned by $(\frac{\partial}{\partial x_1})_p, \cdots, (\frac{\partial}{\partial x_m})_p, (\frac{\partial}{\partial y_1})_p, \cdots, (\frac{\partial}{\partial y_{m+1}})_p$, then f is maximum rank at p, so on sufficiently small neighbourhood of p. We take U, a neighbourhood of p, such that f is a immbedding on U. If we take U sufficiently small, U is diffeomorphic to a real hypersurface of a domain of $\mathbf{C}^{m+1}$. Since f is a diffeomorphism and M is C−R codimension 1, f is a C−R diffeomorphism. q.e.d.

Now we assume $(M,H(M))$ is strongly pseudoconvex. We cover M by finite open sets $U_1, \cdots, U_r$, each U_i can be realized as a real hypersurface S_i of a domain D_i of $\mathbf{C}^{m+1}$. Now we construct a complex manifold with boundary such that a compornent of boundaries is C−R diffeomorphic to $(M,H(M))$.

We take a real valued differentiable function φ_i on D_i, such that $S_i = \{x \in D_i \mid \varphi_i(x) = 0\}$, $d\varphi_i \neq 0$ on S_i and the Levi form of S_i defined by φ_i is positive definite on S_i. Put $D^{\pm} = \{x \in D \mid \varphi_i(x) \gtrless 0\}$. Let $\alpha_i : U_i \longrightarrow S_i$ be the C−R diffeomorphism. We assume $U_1 \cap U_2 \neq \phi$. $\alpha_2 \alpha_1^{-1} : \alpha_1(U_1 \cap U_2) \longrightarrow \alpha_2(U_1 \cap U_2)$ is a C−R diffeomorphism, there exists a domain $V_1 \subset D_1^-$, $V_2 \subset D_2^-$ such that $\partial V_1 \supset U_1$ $\partial V_2 \supset U_2$ and $\alpha_2 \alpha_1^{-1}$ induces a holomorphic diffeomorphism $\alpha_{12} : V_1 \longrightarrow V_2$ (Theorem 2-3).

We take $U_1' \subset\subset U_1$ and $U_2' \subset\subset U_2$ such that $U_1', U_2', U_3, \cdots, U_r$ cover M. Let g_2 be a Riemannian metric on D_2, then $\alpha_{12}{}^* g_2$ is a Riemannian metric on V_1. There exists a Riemannian metric g_1 on D_1 such that g_1 agrees $\alpha_{12}{}^* g_2$ on a neighbourhood of $\overline{\alpha_1(U_1' \cap U_2')}$. Using g_1, g_2, we construct differentiable mappings $\beta_i : U_i \times [0, \epsilon) \longrightarrow D_i$; $i = 1, 2$, for sufficiently small $\epsilon > 0$, by the usual method. Taking ϵ sufficiently small, we may assume $\beta_i(U_1' \cap U_2' \times [0, \epsilon)) \subset V_i$; $i = 1, 2$, then $\alpha_{12}\beta_1 = \beta_2$ on $U_1' \cap U_2' \times [0, \epsilon)$, we can show it easily from the construction of Riemannian metrics g_1, g_2.

We paste $U_1' \times [0, \epsilon)$ and $U_2' \times [0. \epsilon)$ by α_{12}, and obtain a complex manifold V_{12} with boundary such that $U_1' \cup U_2'$ is C−R diffeomorfpic to a part of ∂V_{12}. Repeat the above construction finite time, we obtain

Lemma 3 − 2.

Let $(M,H(M))$ be a strongly pseudoconvex compact C−R submanifold of $\mathbf{C}^n$ of C−R codimension 1. Then there exists a complex manifold V with boundary such that $(M,H(M))$ is C−R diffeomorphic to a compornent of ∂V, Now we assume the C−R dimension of $(M,H(M))$ is greater than 2. Since $(M,H(M))$ is strongly pseudoconvex, there exists a strongly plurisubharmonic function φ on V such that $M=\{x\in V \mid \varphi(x)=0\}$ and $\varphi\leqq 0$ (Gunning-Rossi [4]). Put $V\varepsilon=\{x\in V \mid -\epsilon<\varphi<0\}$ for sufficiently small $\epsilon>0$. Then by the theorem of H. Rossi (H. Rossi [6], Andreotti-Siu [2]), $V\epsilon$ has the Stein completion N, i.e. N is a normal Stein space and $V\epsilon$ is biholomorphic to an open subset of N whose complement is relatively compact in N. Hence we have

Lemma 3 − 3.

Let $(M,H(M))$ be a strongly psudoconvex compact C−R submanifold of C−R dimension $m\geqq 2$ and C−R codimension 1. Then there exists a normal Stein space N with boundary with at most finite singularities such that $(M,H(M))$ is C−R diffeomorphic to the boundary of N.

Since $(M,H(M))$ is strongly pseudoconvex and N is a normal Stein space, C−R functions on M can be continued to holomorphic functions on N by theorem 2−2 and Hartogs-Osgoods theorem. Let $(z_1,\cdots,z_n)$ be a linear coordinate system of $\mathbf{C}^n$. The trace of z_i $(1\leqq i\leqq n)$ on M is a C−R function on M, there exists a holomorphic function $\tilde{z}_i$ on N, differentiable up to the boundary M and $\tilde{z}_i \mid M= z_i$.

Now we consider a mapping $F: N\longrightarrow \mathbf{C}^n$ defined by

$$F(p)=(z_1(p),\cdots,z_n(p)),\ p\in N.$$

It is clear that F is holomorphic on N and C−R diffeomorphic on the boundary.

Lemma 3 − 4.

F is a holomorphic one to one mapping from N into $\mathbf{C}^n$.

Proof. Since F is C−R diffeomorphic on the boundary and holomorphic on N, F is an immedding from a neighbourhood of the boundary into $\mathbf{C}^n$. Then the analytic relation defined by F is proper. By H. Cartan's theorem (H. Cartan [3]), the factor space by the equivalence relation, we denote it by N', has a structure of an analytic space such that the projection $\pi': N\longrightarrow N'$ is holomorphic. Since the analytic relation sepalates the neighbourhood of the boundary, $\pi': N\longrightarrow N'$ is biholomorphic on a neighbourhood of the boundary. We denote the Remmert reduction of N' by N'' and the canonical projection by $\pi'': N'\longrightarrow N''$. Since π' and π'' are biholomorphic on the neighbourhood of the boundary and N', N'' are normal Stein spaces, $\pi''\circ\pi': N\longrightarrow N''$ a is holomorphic diffeomorphism. So $\pi': N\longrightarrow N'$ is one to one, it means F is a one to one mapping. q. e. d.

Main theorem.

Let $(M,H(M))$ be a C−R submanifold of $\mathbf{C}^n$ of C−R dimension m and C−R codimension 1. We asuume M is compact connected strongly pseudoconvex and $m \geqq 2$. Then there exists a Stein subspace of $\mathbf{C}^n$ (not closed), whose boundary is M.

Proof. We shall prove $V=F(N)$ is an analytic space of $\mathbf{C}^n$ (not closed), then V has disiered properties. F is maximal rank on a neighbourhood of the boundary and one to one on N. We take a neighbourhood U of M in $\mathbf{C}^n$ such that

F is an immedding from $F^{-1}(U)$ into $\mathbf{C}^n$. Then $F: N-F^{-1}(U) \longrightarrow \mathbf{C}^n - U$ is a proper holomorphic mapping, so $F(N-F^{-1}(U))$ is an analytic subspace of $\mathbf{C}^n - U$. It is continued to $F(F^{-1}(U))$ analytically, we can show it by taking U sufficiently small, so $V=F(N)$ is an analytic subspace of $\mathbf{C}^n$. q. e. d.

References

1. Andreotti, A., Hill, C. D., *E.E.Levi convexity and the Hans Lewy problem.* Part I. Ann. Scuola Norm. Sup. Pisa. 26, 325-363., Part II. ibid. pp.747−806.
2. Andreotti, A., Siu, Y. T. *Projective embedding of pseudoconcave spaces.* Ann. Scuola Norm. Sup. Pisa, 24, pp. 231−278.
3. Cartan, H. *Quotients of complex analytic spaces.* Tata Inst. Fund. Res. 1960, pp. 1−15.
4. Gunning, R. C., Rossi, H., *Analytic functions of several complex variables.*, Englewood Cliffs, N. J., Prentice-Hall, 1965.
5. Hormander, L., *An introduction to complex analysis in several variables.* Princeton: Van Norstrand 1966.
6. Rossi, H. *Attaching analytic spaces to an analytic space along a pseudoconcave boundary.*, Proc. Conf. Complex Analysis (Minneapolis 1964) Springer-Verlag 1965, pp. 242−256.
7. Rossi, H., *Homogeneous strongly pseudoconvex hypersurfaces.*, Rice Univ. Studies.
8. Sommer, F., *Komplex-analytische Blatterung reeller Mannigfaltigkeiten in* $\mathbf{C}^n$., Math. Ann. Bd 136 pp. 111−133, 1958.
9. Shiga, K. *On holomorphic extension from the boundary.* Nagoya Math. J. Vol. 42 (1971),57−66.
10. Tomassini, G. *Tracce delle funzioni olomorfe sulle sotto varriete analytiche reali d'une varietia complessa.* Ann. Scula Norm. Sup. Pisa 20, 1966.
11. Wells, R. O. Jr., *On the holomorphic hulls of a real submanifold in* $\mathbf{C}^n$. Comm. Pure and Appl. Math. 1966, pp. 145−165.