On the Galois cohomology groups of algebraic tori and Hasse's norm theorem

Kazuo Amano

Dep. of Math., Fac. of Gen. Educ., Gifu Univ.

(Received Oct. 15, 1977)

§1. INTRODUCTION

Let k be an algebraic number field of finite degree and K a finite Galois extension of k with Galois group $\mathfrak{G}$. It is wellknown as the generalization of Hasse's norm theorem that, if we denote by $\widetilde{N}(k)$ the subgroup of the multiplicative group $k^{\times}$ of k consisting of elements which are local norm from K at every places of k, then the group $\widetilde{N}(k)/N_{K/k}K^{\times}$ is isomorphic to a factor group of $\hat{H}^{-3}(\mathfrak{G}, Z)$.

The purpose of the present paper is to generalize this theorem to the case of an algebraic torus T defined over k which splits over K.

§2. GALOIS COHOMOLOGY OF TORI

2.1 Let T_K be the group of K-rational points of T and T_{A_K} the adele group of T over K. We denote by $X=Hom(G_m, T)$ the set of morphisms of G_m into T defined over K and which are also group homomorphisms, where G_m is the multiplicative group of universal domain. We let $\mathfrak{G}$ operate on X by the rule $(s.f)(s.t)=s(f(t))$ for $s \in \mathfrak{G}$, $f \in X$, and $t \in T$. Then it is wellknown that $T_K \cong X \otimes K^{\times}$ and $T_{A_K} \cong X \otimes J_K$ as $\mathfrak{G}$-modules, where J_K is the idele group of K. Denoting by $C_K=J_K/K^{\times}$ the idele class group of K, we have the exact sequence of $\mathfrak{G}$-modules since X is Z-free; $o \longrightarrow X \otimes K^{\times} \longrightarrow X \otimes J_K \longrightarrow X \otimes C_K \longrightarrow o$ and hence we can identify $X \otimes C_K$ with T_{A_K}/T_K as $\mathfrak{G}$-modules. Putting $C_K(T)=T_{A_K}/T_K$, we call $C_K(T)$ the adele class group of T over K. Then the cup multiplication by the canonical class of $\hat{H}^2(\mathfrak{G}, C_K)$ induces an isomorphism $\hat{H}^n(\mathfrak{G}, C) \cong \hat{H}^{n+2}(\mathfrak{G}, C_K(T))$ for every integers n [5].

Analogous result in the local field is the following. Let $\mathfrak{p}$ be a place in k and $\mathfrak{P}$ a place over $\mathfrak{p}$ in K. We denote by $k_{\mathfrak{p}}$ and $K_{\mathfrak{P}}$ the completions of k and K by the places, respectively, and by $\mathfrak{G}_{\mathfrak{P}}$ the Galois group of $K_{\mathfrak{P}}/k_{\mathfrak{p}}$. Then the group $T_{K_{\mathfrak{P}}}$ of $K_{\mathfrak{P}}$-rational points of T is isomorphic to $X \otimes K_{\mathfrak{P}}^{\times}$ as $\mathfrak{G}_{\mathfrak{P}}$-module and the cup multiplication by the canonical class $\alpha_{\mathfrak{P}}$ of $\hat{H}^2(\mathfrak{G}_{\mathfrak{P}}, K_{\mathfrak{P}}^{\times})$ induces an isomorphism $\hat{H}^n(\mathfrak{G}_{\mathfrak{P}}, X) \cong \hat{H}^{n+2}(\mathfrak{G}_{\mathfrak{P}}, T_{K_{\mathfrak{P}}})$ for every integers n [3].

2.2 J. Tate showed the following results in [9]. Let Y be the free abelian group generated by the places $\mathfrak{P}$ of K. An element $s \in \mathfrak{G}$ operates on Y by the rule $s(\sum_{\mathfrak{P}} n_{\mathfrak{P}}\mathfrak{P})=\sum_{\mathfrak{P}} n_{\mathfrak{P}}(s\mathfrak{P})$. We denote by W the kernel of surjective $\mathfrak{G}$-homomorphism $Y \longrightarrow Z$ defined by $\sum_{\mathfrak{P}} n_{\mathfrak{P}}\mathfrak{P} \longrightarrow \sum_{\mathfrak{P}} n_{\mathfrak{P}}$. Then the cup multiplication by the canonical classes $\alpha_1 \in \hat{H}^2(\mathfrak{G}, Hom(Z, C_K))$, $\alpha_2 \in \hat{H}^2(\mathfrak{G}, Hom(Y, J_K))$ and $\alpha_3 \in \hat{H}^2(\mathfrak{G}, Hom(W, K^{\times}))$ gives isomorphisms

$$\hat{H}^n(\mathfrak{G}, X \otimes Z) \cong \hat{H}^{n+2}(\mathfrak{G}, X \otimes C_K),$$
$$\hat{H}^n(\mathfrak{G}, X \otimes Y) \cong \hat{H}^{n+2}(\mathfrak{G}, X \otimes J_K),$$
$$\hat{H}^n(\mathfrak{G}, X \otimes W) \cong \hat{H}^{n+2}(\mathfrak{G}, X \otimes K^\times);$$

moreover there exists a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \hat{H}^n(\mathfrak{G}, X \otimes W) & \longrightarrow & \hat{H}^n(\mathfrak{G}, X \otimes Y) & \longrightarrow & \hat{H}^n(\mathfrak{G}, X \otimes Z) & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & \hat{H}^{n+2}(\mathfrak{G}, X \otimes K^\times) & \longrightarrow & \hat{H}^{n+2}(\mathfrak{G}, X \otimes J_K) & \longrightarrow & \hat{H}^{n+2}(\mathfrak{G}, X \otimes C_K) & \longrightarrow \cdots \end{array}$$

2.3 We recall the semi-local theory which occurs often in the present paper. Let H be a subgroup of a finite group G and $G = \bigcup_g gH$ a left coset decomposition. Let A be a G-module, B an H-module and $A \underset{i}{\overset{j}{\rightleftarrows}} B$ a pair of H-homomorphisms such that $j \circ i$ is the identity on B and $A = \sum_g gi(B)$, direct sum. Then for any G-module M, we have an isomorphism

$$M \otimes A \cong \sum_g M \otimes gi(B)$$
$$\cong \sum_g g(1 \otimes i)(M \otimes B)$$

and the two maps

$$\hat{H}^n(G, M \otimes A) \underset{\mathrm{cor}(1 \otimes i)}{\overset{(1 \otimes j)\mathrm{res}}{\rightleftarrows}} \hat{H}^n(H, M \otimes B)$$

are mutually inverse isomorphisms, where "*res*" denotes restriction map and "cor" denotes corestriction map.

2.4 If $\mathfrak{S}$ is a finite set of places $\mathfrak{p}$ in k, we also denote by the same symbol $\mathfrak{S}$ the set of the places $\mathfrak{P}$ of K which divide some place $\mathfrak{p} \in \mathfrak{S}$. Putting $T_{A_K}^{\mathfrak{S}} = \prod_{\mathfrak{P} \in \mathfrak{S}} T_{K_\mathfrak{P}} \times \prod_{\mathfrak{P} \notin \mathfrak{S}} T_{\mathfrak{o}_\mathfrak{P}}$, the adele group T_{A_K} of T is defined as the inductive limit of $T_{A_K}^{\mathfrak{S}}$ relative to $\mathfrak{S}$, where $T_{\mathfrak{o}_\mathfrak{P}} = X \otimes \mathfrak{U}_\mathfrak{P}$ is the unit group of $K_\mathfrak{P}$. Since $T_{K_\mathfrak{P}} = X \otimes K_{A_K}^{\mathfrak{S}}$, we have $T_{A_K}^{\mathfrak{S}} = \prod_{\mathfrak{p} \in \mathfrak{S}} (\prod_{\mathfrak{P}|\mathfrak{p}} X \otimes K_\mathfrak{P}^\times) \times \prod_{\mathfrak{p} \notin \mathfrak{S}} (\prod_{\mathfrak{P}|\mathfrak{p}} X \otimes \mathfrak{U}_\mathfrak{P})$. For each $\mathfrak{P}$ of K, we denote by $i_\mathfrak{P}: K_\mathfrak{P}^\times \longrightarrow J_K$ the canonical $\mathfrak{G}_\mathfrak{P}$-injection which maps a non zero element of $K_\mathfrak{P}$ onto the idele having that element as $\mathfrak{P}$-component and having 1 as components at all places other than $\mathfrak{P}$. Since $\mathfrak{G}$, $\mathfrak{G}_\mathfrak{P}$, $\prod_{\mathfrak{P}|\mathfrak{p}} K_\mathfrak{P}^\times$ (resp. $\prod_{\mathfrak{P}|\mathfrak{p}} \mathfrak{U}_\mathfrak{P}$), and $K_\mathfrak{P}^\times$(resp. $\mathfrak{U}_\mathfrak{P}$) satisfy the conditions of semi-local theory, we have

$$\hat{H}^n(\mathfrak{G}, X \otimes \prod_{\mathfrak{P}|\mathfrak{p}} K_\mathfrak{P}^\times) \cong \hat{H}^n(\mathfrak{G}_\mathfrak{P}, X \otimes K_\mathfrak{P}^\times),$$
$$\hat{H}^n(\mathfrak{G}, X \otimes \prod_{\mathfrak{P}|\mathfrak{p}} \mathfrak{U}_\mathfrak{P}) \cong \hat{H}^n(\mathfrak{G}_\mathfrak{P}, X \otimes \mathfrak{U}_\mathfrak{P})$$

for any fixed prime $\mathfrak{P}$ dividing $\mathfrak{p}$. Since $\mathfrak{U}_\mathfrak{P}$ is cohomologically trivial if $K_\mathfrak{P}$ is unramified over $k_\mathfrak{p}$, $X \otimes \mathfrak{U}_\mathfrak{P}$ is also cohomologically trivial by the theory of local fields. Therefore, if our set $\mathfrak{S}$ contains all places $\mathfrak{p}$ of k which ramify in K, we have

$$\hat{H}^n(\mathfrak{G},\ T^{\mathfrak{S}}_{A_K}) \cong \prod_{\mathfrak{v}\in\mathfrak{S}} \hat{H}^n(\mathfrak{G},\ \prod_{\mathfrak{P}|\mathfrak{v}} T_{K_\mathfrak{P}})$$

$$\cong \prod_{\mathfrak{v}\in\mathfrak{S}} \hat{H}^n(\mathfrak{G}_\mathfrak{P},\ T_{K_\mathfrak{P}}).$$

Passing to the inductive limit over sufficiently large $\mathfrak{S}$, we have

$$\hat{H}^n(\mathfrak{G},\ T_{A_K}) \cong \varinjlim_{\mathfrak{S}} \hat{H}^n(\mathfrak{G},\ T^{\mathfrak{S}}_{A_K})$$

$$\cong \sum_{\mathfrak{v}} \hat{H}^n(\mathfrak{G}_\mathfrak{P},\ T_{K_\mathfrak{P}}).$$

Therefore we have the following

PROPOSITION 1. $\hat{H}^n(\mathfrak{G},\ T_{A_K}) \cong \sum_{\mathfrak{v}} \hat{H}^n(\mathfrak{G}_\mathfrak{P},\ T_{K_\mathfrak{P}})$ *for every integers* n.

2.5 Putting $Y_\mathfrak{v} = \sum_{\mathfrak{P}|\mathfrak{v}} Z\mathfrak{P}$, we have $Y = \sum_{\mathfrak{v}} Y_\mathfrak{v}$ (direct) as $\mathfrak{G}$-module. Accordingly we have $X \otimes Y = \sum_{\mathfrak{v}} (X \otimes Y_\mathfrak{v})$ as $\mathfrak{G}$-module. For each place $\mathfrak{P}$ of K, we define a $\mathfrak{G}_\mathfrak{P}$-homomorphism $i'_\mathfrak{P}$: $Z \longrightarrow Y$ by $i'_\mathfrak{P}(n) = n\mathfrak{P}$. Since $\mathfrak{G}$, $\mathfrak{G}_\mathfrak{P}$, $Y_\mathfrak{v}$, and $Z\mathfrak{P}$ satisfy the the conditions of semi-local theory, we have

$$\hat{H}^n(\mathfrak{G},\ X \otimes Y) \cong \sum_{\mathfrak{v}} \hat{H}^n(\mathfrak{G},\ X \otimes Y_\mathfrak{v})$$

$$\cong \sum_{\mathfrak{v}} \hat{H}^n(\mathfrak{G}_\mathfrak{P},\ X).$$

Therefore we have the following

PROPOSITION 2. $\hat{H}^n(\mathfrak{G},\ X \otimes Y) = \sum_{\mathfrak{v}} \hat{H}^n(\mathfrak{G}_\mathfrak{P},\ X)$ *for every integers* n.

2.6 Using these propositions, we have the following

PROPOSITION 3. *The following diagram is commutative*:

$$\begin{array}{ccc}
\sum_{\mathfrak{v}} \hat{H}^n(\mathfrak{G}_\mathfrak{P},\ X) & \xrightarrow[\sim]{i'} & \hat{H}^n(\mathfrak{G},\ X \otimes Y) \\
\downarrow \sum_{\mathfrak{P}} \cup \alpha_\mathfrak{v} & & \downarrow \cup \alpha_2 \\
\sum_{\mathfrak{v}} \hat{H}^{n+2}(\mathfrak{G}_\mathfrak{P},\ T_{K_\mathfrak{P}}) & \xrightarrow[\sim]{i} & \hat{H}^{n+2}(\mathfrak{G},\ T_{A_K})
\end{array}$$

PROOF. The top horizontal isomorphism i' is induced by the maps

$$\hat{H}^n(\mathfrak{G}_\mathfrak{P},\ X) \xrightarrow{\mathrm{cor}(1 \otimes i'_\mathfrak{P})} \hat{H}^n(\mathfrak{G},\ X \otimes Y),$$

and the bottom horizontal isomorphism i is induced by the maps

$$\hat{H}^{n+2}(\mathfrak{G}_\mathfrak{P},\ T_{K_\mathfrak{P}}) \xrightarrow{cor(1 \otimes i_\mathfrak{P})} \hat{H}^{n+2}(\mathfrak{G},\ T_{A_K}).$$

By the fundamental relation between corestriction and cup product, we have

$$\begin{aligned}
&\alpha_2 \cup (cor(1 \otimes i'_\mathfrak{P})\ \xi) = cor(res\ \alpha_2 \cup (1 \otimes i'_\mathfrak{P})\ \xi) \\
&= cor(j_\mathfrak{P} \cdot res\ \alpha_2 \cup \xi) = cor(i_\mathfrak{P}\, \alpha_\mathfrak{P} \cup \xi) \\
&= cor(1 \otimes i_\mathfrak{P})(\alpha_\mathfrak{P} \cup \xi),
\end{aligned}$$

where $\xi \in \hat{H}^n(\mathfrak{G}_\mathfrak{P},\ X)$, and $j_\mathfrak{P}$(*) is the projection $Hom(Y,\ X) \longrightarrow X$ defined by $j_\mathfrak{P}(f) = f(\mathfrak{P})$ for $\mathfrak{P} \in \mathfrak{S}$. Therefore we have our proposition.

(*) By Tate's paper [8], $j_\mathfrak{P}$ has the property $j_\mathfrak{P}(\mathrm{res}\ \alpha_2) = i_\mathfrak{P}\ \alpha_\mathfrak{P}$.

§3. Hasse's Norm Theorem of Tori

We consider the following exact sequence of $\mathfrak{G}$-modules

$$0 \longrightarrow T_K \xrightarrow{i} T_{A_K} \xrightarrow{j} C_K(T) \longrightarrow 0$$

Passing to cohomology groups, we have the exact sequence

$$\cdots \longrightarrow \hat{H}^{-1}(\mathfrak{G}, T_K) \xrightarrow{i^*} \hat{H}^{-1}(\mathfrak{G}, T_{A_K}) \xrightarrow{i^*} \hat{H}^{-1}(\mathfrak{G}, C_K(T))$$
$$\longrightarrow \hat{H}^0(\mathfrak{G}, T_K) \xrightarrow{j^*} \hat{H}^0(\mathfrak{G}, T_{A_K}) \xrightarrow{j^*} \hat{H}^0(\mathfrak{G}, C_K(T)) \longrightarrow \cdots.$$

Proposition 4. *The following diagram is commutative*

$$\begin{array}{ccccc} \sum_{\mathfrak{p}} \hat{H}^n(\mathfrak{G}_{\mathfrak{P}}, X) & \xrightarrow{\sum_{\mathfrak{P}} \alpha_{\mathfrak{P}}} & \sum_{\mathfrak{p}} \hat{H}^{n+2}(\mathfrak{G}_{\mathfrak{P}}, T_{K_{\mathfrak{P}}}) & \xrightarrow{\sim} & \hat{H}^{n+2}(\mathfrak{G}, T_{A_K}) \\ \downarrow \sum_{\mathfrak{p}} \zeta_{\mathfrak{P}} & & & & \downarrow j^* \\ \hat{H}^n(\mathfrak{G}, X) & & \xrightarrow{\cup \alpha_1} & & \hat{H}^{n+2}(\mathfrak{G}, C_K(T)) \end{array}$$

Proof. By virtue of Tate's commutative diagram 2.2 and proposition 3, we have the following commutative diagram

$$\begin{array}{ccccc} \sum_{\mathfrak{p}} \zeta_{\mathfrak{P}} : \sum_{\mathfrak{p}} \hat{H}^n(\mathfrak{G}_{\mathfrak{P}}, X) & \longrightarrow & \hat{H}^n(\mathfrak{G}, X \otimes Y) & \longrightarrow & \hat{H}^n(\mathfrak{G}, X) \\ \downarrow \sum_{\mathfrak{P}} \alpha_{\mathfrak{P}} & & \downarrow \cup \alpha_2 & & \downarrow \cup \alpha_1 \\ \sum_{\mathfrak{p}} \hat{H}^{n+2}(\mathfrak{G}_{\mathfrak{P}}, T_{K_{\mathfrak{P}}}) & \xrightarrow{\sim} & \hat{H}^{n+2}(\mathfrak{G}, T_{A_K}) & \xrightarrow{j^*} & \hat{H}^{n+2}(\mathfrak{G}, C_K(T)) \end{array}$$

This proves the proposition

Denote by $\tilde{N}(T) = T_k \cap (\bigcap_{\mathfrak{p}} N_{K_{\mathfrak{P}}/k_{\mathfrak{p}}} T_{K_{\mathfrak{P}}})$ the subgroup of T_k consisting of elements which are local norm from T_K at every places of k, where T_k is the group of k-rational points of T. Then, as our main result, we have the generalization of Hasse's norm theorem to the case of an algebraic torus.

Theorem. *Let F be the subgroup of $\hat{H}^{-3}(\mathfrak{G}, X)$ generated by $\zeta_{\mathfrak{P}}(\hat{H}^{-3}(\mathfrak{G}_{\mathfrak{P}}, X))$ for every $\mathfrak{p}$. Then we have an isomorphism*

$$\tilde{N}(T)/N_{K/k}\, T_K \cong \hat{H}^{-3}(\mathfrak{G}, X)/F.$$

Proof. By virtue of proposition 4, we have

$$\begin{aligned} \tilde{N}(T)/N_{K/k}\, T_K &= Ker(i^*) \\ &\cong \hat{H}^{-1}(\mathfrak{G}, C_K(T))/j^*(\hat{H}^{-1}(\mathfrak{G}, T_{A_K})) \\ &\cong \hat{H}^{-3}(\mathfrak{G}, X)/F. \end{aligned}$$

Corollary ([5]). *If K/k is cyclic extension, we have $\tilde{N}(T) = N_{K/k}\, T_K$.*

Proof. By virtue of Kneser's paper [3], there is, for every integers n, a canonical injection

$$\hat{H}^n(\mathfrak{G}, T_K) \longrightarrow \sum_{\mathfrak{p}} \hat{H}^n(\mathfrak{G}_{\mathfrak{P}}, T_{K_{\mathfrak{P}}}).$$

Therefore we have $Ker(i^*) = 0$ by proposition 1.

§4. Application to Non-Galois Extensions

In this section, we give Hasse's norm theorem to the case of non-Galois extensions using above commutative diagram for Tasaka's special tori [8].

4.1 Let L be a separable extension of k and K a finite Galois extension of k containing L. We denote by $\mathfrak{G}$ and $\mathfrak{H}$ the Galois group of K/k and K/L, respectively. Let $\mathfrak{G}=\bigcup_{g} g\mathfrak{H}$ be a left coset decomposition. We consider the following left $\mathfrak{G}$-modules

$$\Lambda = Z[\mathfrak{G}/\mathfrak{H}],\quad R=\Lambda/Zu,$$

where $Z[\mathfrak{G}/\mathfrak{H}]=\sum_{a} Za$, $a=g\mathfrak{H}$, and $u=\sum_{a} a$. To the *Z-free* $\mathfrak{G}$-module R, there is a corresponding torus T which the module R is the character module $Hom(T, G_m)$. Then T. Tasaka [8] proved the following isomorphisms

$$\hat{H}^1(\mathfrak{G}, T_K)\cong k^*/N_{L/k}L^\times,$$

$$\hat{H}^1(\mathfrak{G}, T_{A_K})\cong J_k/N_{L/k}J_L,$$

where J_k and J_L are the idele groups of k and L, respectively.

Now we consider the exact sequence of $\mathfrak{G}$-modules:

$$0\longrightarrow T_A \xrightarrow{\ i\ } T_{A_K}\xrightarrow{\ j\ } C_K(T)\longrightarrow 0$$

Passing to cohomology groups, we obtain the following exact sequence:

$$\cdots\longrightarrow \hat{H}^0(\mathfrak{G}, T_{A_K})\xrightarrow{\ j^*\ }\hat{H}^0(\mathfrak{G}, C_K(T))\longrightarrow \hat{H}^1(\mathfrak{G}, T_K)\xrightarrow{\ i^*\ }\hat{H}^1(\mathfrak{G}, T_{A_K})\longrightarrow\cdots$$

By virtue of the commutative diagram of proposition 4 and above isomorphisms, we have the following

Theorem. *Let $\tilde{N}(k)$ be the subgroup of the multiplicative group $k^\times$ of k consisting of elements which are local norm from L at every places of k. Then we have an isomorphism*

$$\tilde{N}(k)/N_{L/k}L^*\cong \hat{H}^{-2}(\mathfrak{G}, X)/F.$$

where *F denotes the subgroup of $\hat{H}^{-2}(\mathfrak{G}, X)$ generated by $\zeta_{\mathfrak{P}}(\hat{H}^{-2}(\mathfrak{G}_{\mathfrak{P}}, X))$ for every $\mathfrak{p}$.*

4.2 Using above theorem, we give an example such that Hasse's norm theorem is valid [8]. Let L be a separable non-cyclic cubic extension of k, and $\mathfrak{G}=\{\sigma^3=\tau^2=1,\ \tau^{-1}\sigma\tau=\sigma^2\}$ the Galois group of the Galois extension K/k. We consider a two-dimensional torus

$$T=\{t\in R_{K/k}(G_m)\mid t^{1+\sigma+\sigma^2}=1,\ t^\tau=t\},$$

where $R_{K/k}(G_m)$ donotes the algebraic group defined over k obtained by restricting the field of definition K to k. By [10], we have $\hat{H}^{-2}(\mathfrak{G}, X)=0$, and hence we obtain Hasse's norm theorem for L/k.

References

[1] E. Artin and J. Tate, Class field theory, Benjamin, 1967.
[2] C. Chevalley, Class field theory, Nagoya, 1953-54.
[3] M. Kneser, Lecture on Galois cohomology of classical groups, Tata Inst. Fund. Res., Bombay,

1969.
〔4〕 T. ONO, Arithmetic of algebraic tori, Ann. of Math., 74(1961), 101-139.
〔5〕 T. ONO, On the Tamagawa number of algebraic tori, Ann. of Math., 78(1963), 47-74.
〔6〕 J.-P. SERRE, Corps locaux, Hermann Paris, 1962.
〔7〕 T. TASAKA, On the second cohomology groups of the fundamental groups of simple algebraic groups over perfect fields, J. Math. Soc. Japan, 21(1969), 244-258.
〔8〕 T. TASAKA, Remarks on the validity of Hasse's norm theorem, J. Math. Soc. Japan, 22(1970), 330-341.
〔9〕 J. TATE, The cohomology groups of tori in finite Galois extensions of number fields, Nagoya Math. J., 27(1966), 709-719.
〔11〕 V. E. VOSKRESENSKII, On two-dimensional algebraic tori, Amer. Math. Soc. Transl., 73(1968), 190-195.