A remark on a complete curve in complex Euclidean space.

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§1. *Introduction*

One of the most interesting problems in the theory of complex submanifolds in a complex Euclidean space C^n is to study the global behavior of curvature of submanifolds. A fundumental question is

Question I. Does there exist a complete complex submanifold of C^n with holomorphic sectional curvature bounded from above by a negative constant ?

T. Sasaki and K. Shiga [2] and P. Yang [3] have already investigated this problem, and they gave a complete answer in the case of codimension one.

In this short note, we shall study the relationship between the notion of completeness and the notion of closedness. Of course a complete submanifold of C^n is not necessarily a closed submanifold (Example). So we shall consider the following problem for the present.

Question II. Does there exist a complete submanifold of C^n which has a bounded image ?

This problem is originally raised by S. S. Chern [1] for minimal submanifolds in R^n. P. Yang showed that there exists some relation between Question I and Question II. But Qusetion I is not solved in general, so we shall study Question II directly. The following is the main theorem of this note.

THEOREM. Let C *be a complete curve in* C^n, *and* κ *be the Gaussian curvature of* C. *If there exists a positive constant* k *such that* $-k<\kappa\leqq 0$, C *is not bounded.*

I have received many suggestions through the conversations from Professor T. Sasaki and Professor S. Takeuchi. I would like to express my cordial thanks to them.

§2. *Proof of the theorem and an example.*

By a curve C in C^n we will mean a holomorphic immersion $\zeta: C \rightarrow C^n$, where C is an open Riemann surface. Let ds^2 be the canonical Kähler metric on C^n. We will say that C is a complete curve in C^n if the induced metric $\zeta^* ds^2$ is a complete metric on C.

We may assume that C is simply connected by considering the uiversal covering space if necessary. Then C is biholomorphic to $\mathbf{C}$ or the unit disk $D=\{z\varepsilon\mathbf{C} \mid |z|<1\}$. If C is biholomorphic to $\mathbf{C}$, there exists no immersion of $\mathbf{C}$ to C^n with bounded image, since there exists no nontrivial bounded holomorphic function on $\mathbf{C}$.

Then we consider the case that $\zeta: D \rightarrow C^n$ is a holomorphic immersion such that

ζ^*ds^2 is a complete metric with curvature bounded from below by a negative constant.

Put $\zeta^*ds^2=h\ dzd\bar{z}$, then h is a positive real analytic function on D. Since ζ^*ds^2 is a complete metric with curvature bounded from below, there exists a positive constant c by Yau's generalized Schwarz lemma ([4]) such that

$$\zeta^*ds^2 \geqq c\ ds^2{}_D, \tag{1}$$

where $ds^2{}_D$ is the Poincaré metric on D, i.e.,

$$ds^2{}_D=\frac{dzd\bar{z}}{(1-|z|^2)^2}\ . \tag{2}$$

Then by (1) and (2)

$$h \geqq c\ (1-|z|^2)^{-2} \tag{3}$$

We define a C^∞ function $\gamma(z)$ on D by

$$\gamma(z)=\|\zeta(z)-\zeta(p)\|^2, \tag{4}$$

where p is a fixed point of D and $\|\cdot\|$ is the Euclidean norm in C^n.

Let $\triangle=1/h\dfrac{\partial^2}{\partial z\partial\bar{z}}$ be the Laplacian on D. Then $\gamma(z)$ satisfies $\triangle\gamma=1$. (5)

We will show that $\gamma(z)$ is not a bounded function on D.
For $0<r<1$, we put

$$M(r)=\frac{1}{2\pi}\int_0^{2\pi}\gamma(re^{i\theta})\ d\theta \tag{6}$$

Then for $0<r_0<r<1$,

$$M(r)-M(r_0)=\int_{r_0}^{r}\frac{d}{dr}M(r)\ dr$$

$$=\int_{r_0}^{r}\frac{1}{2\pi}\int_0^{2\pi}\frac{d}{dr}\gamma(re^{i\theta})\ d\theta dr \qquad \text{(by (6))}$$

$$=\int_{r_0}^{r}\frac{1}{2\pi r}\int_0^{2\pi}\frac{d}{dr}\gamma(re^{i\theta})\ r\ d\theta dr$$

$$=\int_{r_0}^{r}\frac{1}{2\pi r}\iint_{D_r}\widetilde{\triangle}\ \gamma dxdy$$

by Green's formula, where $\widetilde{\triangle}=4\dfrac{\partial^2}{\partial z\partial\bar{z}}$ and $D_r=\{z\in C\,|\,|z|<r\}$,

$$=\int_{r_0}^{r}\frac{4}{2\pi r}\iint_{D_r}h\ dxdy \qquad \text{(by (5))}.$$

It means by (3), that

$$M(r)-M(r_0)\geqq\int_{r_0}^{r}\frac{2}{\pi r}\iint_{D_r}\frac{1}{(1-r^2)^2}dxdy$$

Calculating the right hand side, we obtain the following inequality

$$M(r)-M(r_0)\geqq C_1\ \{\log r-\frac{1}{2}\log(1-r)-\frac{1}{2}\log(1+r)\}-C_2\ ,$$

where C_1 and C_2 are positive constants independent of r.
The right hand side tends to infinity if r tends to 1. It means

$$\lim_{r\to 1}M(r)=\infty.$$

This means that $\gamma(z)$ is not bounded, and it completes the proof of the theorem.

There exists an example that a complete curve in C^n is not necessarily closed.

EXAMPLE. Let (z, w) be the linear coordinate system C^2. Let S de a closed curve defined by

$$w = \frac{1}{z(z-1)}$$

Since S is biholomorphic to $C - \{0, 1\}$, the universal covering space is the unit disk D. Then there exists a covering map $\psi: D \to S$. We define an imbedding $\psi: D \to C^3$ by

$$\zeta(z) = (z, \ \psi(z)).$$

Since S is a closed submanifold, S is a closed curve. From the construction

$$\zeta^* ds^2 \geqq \psi^* ds^2,$$

where ds^2 in the left hand side and in the right hand side mean the canonical metric of C^3 and C^2 respectivery. It means that $\zeta: D \to C^3$ is a complete curve in C^3. For a point $p \in S$, $\psi^{-1}(p)$ is an infinite set then $\zeta(D)$ is not a closed submanifold of C^3.

We are interested in obtaining under what condition a complete submanifold is a closed submanifold. But it seems to be a very difficult problem.

References

〔1〕 Chern,S,S. The geometry of G-structures, Bull. A.M.S. 72(1966), 167-219.

〔2〕 Sasaki, T. and Shiga, K. Remarks on the curvature of hypersurfaces in C^n, (to appear).

〔3〕 Yang, P. Curvature of complex submanifolds of C^n, J. Diff. Geom. (to appear).

〔4〕 Yau, S. T. A general Schwarz lemma for Kähler manifolds, (preprint).