Smoothness of Roots of Hyperbolic Polynomials with respect to One-Dimensional Parameter

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§1. Introduction

In this article, we consider an algebraic equation whose coefficients have one-dimensional parameter t. In general, the roots of such an equation, as functions of t, have no differentiability, however smooth the coefficients may be. If we assume, however, that the equation has only real roots, then we can prove some smoothness with suitable smoothness assumptions on the coeffients.

Definition. A polynomial of X, $P(X) = X^m + b_1 X^{m-1} + \cdots + b_m$ ($b_j \in \boldsymbol{R}$), is called *hyperbolic* if the equation $P(X) = 0$ has only real roots.

M. D. Bronshtein ([1]) proved the following theorem that played an important role in constructing parametrices of weakly hyperbolic operators with Gevrey coefficients ([2]). (For simplicity, we omit the secondary parameters.)

Theorem A. *Let $P(t; X) = X^m + B_1(t) X^{m-1} + \cdots + B_m(t)$ be hyperbolic for any $t \in I = (a, b)$. Assume that the multiplicity of its roots does not exceed r and $B_j \in C^r(I)$ ($j = 1, \cdots, m$). Then, the followings hold.*

(1) *For any $t_0 \in I$, there exist m roots $\lambda_j(t_0; t)$ ($j = 1, \cdots, m$) of $P(t; X) = 0$ such that $\lambda_j(t_0; t)$ are differentiable at $t = t_0$.*

(2) *For any compact set $K \subset I$, the set $\{\lambda_j'(t_0; t_0);\ t_0 \in K,\ j = 1, \cdots, m\}$ is bounded.*

Note that the differentiability of $\lambda_j(t_0; t)$ is assured only at $t = t_0$. It is natural to ask whether we can take roots that is differentiable on I or not. Moreover, we want to know whether we can take C^1-roots or not. The aim of this article is to give some answers to these questions.

In the followings, we use the notation $f'(t_0 \pm 0) = \lim_{t \to t_0 \pm 0} f'(t)$ and $f'_\pm(t_0) = \lim_{t \to t_0 \pm 0} \dfrac{f(t) - f(t_0)}{t - t_0}$ (right and left derivatives).

§2. Statement of the Results and Remarks

Theorem B. *Assume the same assumptions as Theorem A. Then, the followings hold.*

(1) *If $\lambda(t) \in C^0(I)$ and $P(t; \lambda(t)) = 0$ on I, then for any $t_0 \in I$, there exist $\lambda'_\pm(t_0)$. Further, for any compact set $K \subset I$, the set $\{\lambda'_\pm(t);\ t \in K\}$ is bounded.*

(2) *Let the multiplicity of $X = \lambda(t_0)$ be q. If $\lambda_j(t) \in C^0(I)$, $\lambda_j(t_0) = \lambda(t_0)$ ($j = 1, \cdots, q$) and $P(t; X)$ is divisible by $(X - \lambda_1(t)) \cdots (X - \lambda_q(t))$ as a polynomial of X, then the sets*

$D_{\pm} = \{\lambda'_{j,\pm}(t_0)\,;\, j = 1, \cdots, q\}$ *are respectively the roots of the same equation* $a_0X^q + a_1X^{q-1} + \cdots + a_q = 0$, *where* $a_j = \partial_t^j \partial_X^{q-j} P(t_0\,;\,\lambda(t_0)) / (q-j)!j!$ $(j = 0, 1, \cdots, q)$.

(3) *There exist* $\lambda_1, \cdots, \lambda_m \in C^0(I)$ *such that* $P(t\,;X) = (X-\lambda_1(t))\cdots(X-\lambda_m(t))$ *and* λ_j $(j=1, \cdots, m)$ *are differentiable on* I.

(4) *Assume that* $B_j \in C^{2r}(I)$ $(j=1, \cdots, m)$, *furthermore. If* $\lambda \in C^0(I)$ *is differentiable on* I *and* $P(t\,;\lambda(t))=0$ *on* I, *then* $\lambda \in C^1(I)$.

REMARKS. (i) The claims (1) and (2) are another expression of what M. D. Bronshtein has really proved in [1]. (Hence, we shall omit the proof.) The most important part is the boundedness of the set $\{\lambda'_{\pm}(t)\,;\, t \in K\}$. The other part of (1), (2) had essentially been known before Bronshtein. In fact, the key point of its proof is the fact that if $B_k(t_0) \neq 0$, $B_j(t_0) = 0 (j>k)$, then $B_j^{(i)}(t_0) = 0 (j>k,\, i<j-k)$, and this fact was used, for example, by L. Hörmander ([5]).

(ii) Even if we assume that $m=2$ and $B_j \in C^\infty(I)$, we can not take C^2-roots generally. In fact, there exists a non-negative C^∞-function $f(t)$ on $[-1, 1]$ such that $f(0)=0$, $f(t) > 0$ for $t \neq 0$ and $[\sqrt{f(t)}]''$ is unbounded on $(0, 1)$ (G. Glaeser [4]).

(iii) When $r=1$ or 2, we can omit the assumption that $B_j \in C^{2r}(I)$ $(j=1, \cdots, m)$ in (4). (When $r=1$, it is trivial. When $r=2$, the proof is given in §4.) The author does not know whether we can also omit this assumption or not when $r \geqq 3$.

§3. Proof of Theorem B

We begin with two lemmas, which reduce the proof to a simpler situation. (Lemma 2 is well-known, hence its proof is omitted.)

LEMMA 1. *Assume the same assumptions as Theorem A. Take* $c \in I$ *and put* $I_1 = (a, c)$, $I_2 = (c, b)$. *If there exist* $\lambda_j^\sim(t) \in C^0(I_1 \cup I_2)$ $(j=1, \cdots, m)$ *such that* $\lambda_j^\sim$ *are differentiable on* $I_1 \cup I_2$ *and* $P(t; X) = (X-\lambda_1^\sim(t))\cdots(X-\lambda_m^\sim(t))$ *on* $I_1 \cup I_2$, *then there exist* $\lambda_j \in C^0(I)$ $(j=1, \cdots, m)$ *such that* λ_j *are differentiable on* I *and* $P(t\,;X) = (X-\lambda_1(t))\cdots(X-\lambda_m(t))$ *on* I.

PROOF. Put $I_1^\sim = (a, c]$, $I_2^\sim = [c, b)$. We can extend $\lambda_j^\sim$ as $\lambda_{j,k}^\sim \in C^0(I_k^\sim)$ $(k=1, 2\,; j=1, \cdots, m)$. By (1) of Theorem B, there exist $\lambda'^\sim_{j,1,-}(c)$ and $\lambda'^\sim_{j,2,+}(c)$ $(j=1, \cdots, m)$, By (2) of Theorem B, these are the roots of the same equations, hence we can renumber these roots such that $\lambda_{j,1}^\sim(c) = \lambda_{j,2}^\sim(c)$ and $\lambda'^\sim_{j,1,-}(c) = \lambda'^\sim_{j,2,+}(c)$ $(j=1, \cdots, m)$. Thus, if we connect these $\lambda_{j,k}^\sim$ at $t=c$, then we can take $\lambda_j \in C^0(I)$ such that λ_j are differentiable on I.

LEMMA 2. *Consider* $P(t; X) = X^m + A_1(t)X^{m-1} + \cdots + A_m(t)$, *where* $A_j \in C^s(I)$ $(0 \leqq s \leqq \infty)$. *If* $p(t_0\,; X) = 0$ *has the distinct roots* $x_1, \cdots, x_k$ *and the multiplicity of* x_j *is* m_j $(j=1, \cdots, k)$, *then there exist a positive* ε *and polynomials* $p_j(t\,; X) = X^{m_j} + A_{j,1}(t)X^{m_j-1} + \cdots + A_{j,m_j}(t)$ $(j=1, \cdots, k)$ *such that* $A_{j,i} \in C^s(t_0-\varepsilon, t_0+\varepsilon)$ $(1 \leqq i \leqq m_j,\, 1 \leqq j \leqq k)$, $p_j(t_0; X) = 0$ *has the unique root* x_j *with multiplicity* m_j *and* $p(t\,; X) = p_1(t\,; X)\cdots p_k(t\,; X)$ *on* $(t_0-\varepsilon, t_0+\varepsilon)$.

First, we prove (3) by the induction on r. If $r=1$, then the result is trivial. (Any continuous root is C^1.) Now, assume that (3) is valid if $r\leqq k-1$ $(k\geqq 2)$ and that the multiplicity of the roots of P is not larger than k. By Lemma 1, we have only to show that for any $t_0\in I$, there exists $\varepsilon>0$ such that on $(t_0-\varepsilon, t_0+\varepsilon)$, instead of on I, (3) hold. By Lemma 2, we may assume $k=m$ without loss of generality. Further, we may assume $B_1(t)\equiv 0$ by the transformation $X'=X+B_1(t)/m$.

Put $J=\{t\in I\ ; P(t;X)=0$ has the unique root 0 (with the multiplicity m)$\}$ and $I_1=I\setminus J$. Note that J is closed and I_1 is open. Since $I_1=\bigcup_{n=1}^{\infty} I^{(n)}$ where $I^{(n)}$ are disjoint open intervals, by the assumption of the induction, we can take the roots of $P(t;X)=0$ as $\lambda_j^-(t)\in C^0(I_1)$ $(j=1,\cdots,m)$ which are differentiable on I_1. Put $J_1=\{t\in J; t$ is isolated in $J\}$, $J_2=J\setminus J_1$ and $I_2=I_1\cup J_1$. Note that J_1 is at most countable. By Lemma 1, we can take the roots of $P(t;X)=0$ as $\lambda_j\in C^0(I_2)$ which are differentiable on I_2. Put $\lambda_j(t)=0$ if $t\in J_2$. We shall show that these λ_j are also differentiable at t_0 for any $t_0\in J_2$. Assume that there exist $t_n\in J$ $(n=1, 2,\cdots)$ such that $t_n>t_0$ and $t_n\to t_0$ $(n\to\infty)$. Since $\lambda_j(t)$ is right differentiable at t_0 and $\lambda_j(t_n)=0$, we have $\lambda'_{j,+}(t_0)=0$ $(j=1,\cdots,m)$. By (2) of Theorem B, we have $\lambda'_{j,-}(t_0)=0$ $(j=1,\cdots,m)$, hence λ_j are differentiable at t_0 and $\lambda_j'(t_0)=0$. The same arguments go well if there exist $t_n\in J$ such that $t_n<t_0$ and $t_n\to t_0$ $(n\to\infty)$. Thus, λ_j are differentiable on I.

Next, we prove (4) also by the induction on r. By a similar (and simpler) arguments as above, we may assume that (4) is valid if $r\leqq k-1$ $(k\geqq 2)$ and that $k=m$, $B_1(t)\equiv 0$. Put J and I_1 as above. By the assumption of the induction, we have $\lambda_j\in C^1(I_1)$ $(j=1,\cdots, m)$. Take an arbitrary $t_0\in J$. Note that $\lambda_j(t_0)=0$ $(j=1,\cdots,m)$ and $B_j(t)=(t-t_0)^j\tilde{B}_j(t)$, where $\tilde{B}_j\in C^m(I)$ (see Remarks (i)). By the transformation $X=(t-t_0)X'$, the equation $P(t;X)=0$ is transformed into $(t-t_0)^m Q(t;X')=0$, where $Q(t;X')=X'^m+\tilde{B}_2(t)X'^{m-2}+\cdots+\tilde{B}_m(t)$. Hence, $\mu_j(t)=\lambda_j(t)/(t-t_0)$ $(j=1,\cdots,m)$ are the roots of $Q(t;X)=0$ $(t\neq t_0)$. By (1) of Theorem B, the set $\{\mu_j'(t)\,;\, j=1,\cdots,m\}$ is bounded near $t=t_0$, that is, there exist $\varepsilon>0$ and $M>0$ such that $|\mu_j'(t)|\leqq M$ if $0<|t-t_0|<\varepsilon$. Since $\mu_j'(t)=\{\lambda_j'(t)(t-t_0)-\lambda_j(t)\}/(t-t_0)^2$ $(t\neq t_0)$, we have $|\lambda_j'(t)-\lambda_j(t)(t-t_0)^{-1}|\leqq M|t-t_0|$ if $0<|t-t_0|<\varepsilon$. Hence, there holds $\lim_{t\to t_0}\lambda_j'(t)=\lambda_j'(t_0)$. Thus, we have $\lambda_j\in C^1(I)$.

§4. Proof of Remarks (iii)

By the same argument in §3, we may assume that $r=m$, that is, we have only to prove the following proposition.

Proposition. *Consider $P(t;X)=X^2-A(t)X-B(t)=0$, where $A, B\in C^2(I)$ and $A(t)^2+4B(t)\geqq 0$. If $\lambda\in C^0(I)$ is differentiable on I and $P(t;\lambda(t))=0$ on I, then $\lambda\in C^1(I)$.*

Proof. By the transformation $X'=X-A(t)/2$, we may assume that $A(t)\equiv 0$ and $B(t)\geqq 0$ on I. Put $J=\{t\in I\,;\, B(t)\doteq 0\}$ and $I_1=I\setminus J$. Note that $\lambda\in C^1(I_1)$, $\lambda(t)=0$ on J and $B(t)=B'(t)=0$ on J. Take an arbitrary $t_0\in J$. Since $B(t_0)=B'(t_0)=0$, we can write

$B(t)=(t-t_0)^2 B^{\sim}(t)$ where $B^{\sim}\in C^0(I)$ and $B^{\sim}(t_0)=B''(t_0)/2$. Hence $\lambda'(t_0)=\pm\sqrt{B^{\sim}(t_0)}$ $=\pm\sqrt{B''(t_0)/2}$. We may assume that $\lambda'(t_0)=\sqrt{B^{\sim}(t_0)}$. We have only to prove that $\lambda'(t)\to\lambda'(t_0)$ $(t\to t_0)$.

Case 1. Assume that $B^{\sim}(t_0)\neq 0$. There exists a positive ε such that $B^{\sim}(t)>0$ if $|t-t_0|<\varepsilon$. Hence, by $\lambda'(t_0)=\sqrt{B^{\sim}(t_0)}$, we have $\lambda(t)=(t-t_0)\sqrt{B^{\sim}(t)}$ if $|t-t_0|<\varepsilon$. Since $\sqrt{B^{\sim}(t)}$ is differentiable if $0<|t-t_0|<\varepsilon$, we have $\lambda'(t)=\sqrt{B^{\sim}(t)}+(t-t_0)B^{\sim\prime}(t)/2\sqrt{B^{\sim}(t)}$. On the other hand, by $B'(t)=2(t-t_0)B^{\sim}(t)+(t-t_0)^2B^{\sim\prime}(t)$, we have $(t-t_0)B^{\sim\prime}(t)=$ $B'(t)/(t-t_0)-2B^{\sim}(t)\to B''(t_0)-2B^{\sim}(t_0)=0$ $(t\to t_0)$. Thus, $\lambda'(t)\to\sqrt{B^{\sim}(t_0)}=\lambda'(t_0)$ $(t\to t_0)$.

Case 2. Assume that $B^{\sim}(t_0)=0$. Since $\sqrt{B^{\sim}(t)}\to 0$ $(t\to t_0)$, the above argument is violated. We use the following well-known lemma. (See, for example, J. Dieudonné [3].)

LEMMA 3. *Assume that* $f\in C^2(-2\varepsilon,2\varepsilon)$ $(\varepsilon>0)$, $f(x)\geqq 0$ *on* $(-2\varepsilon,2\varepsilon)$ *and* $f(0)=f'(0)=f''(0)=0$. *There holds* $|f'(x)|^2\leqq 2\left(\sup_{|x|<2\varepsilon}|f''(x)|\right)f(x)$ *if* $|x|\leqq\varepsilon$.

We have already known that $\lambda'(t)=\pm\sqrt{B''(t)/2}$ on J and $\lambda'(t)=\pm B'(t)/2\sqrt{B(t)}$ on I_1. By Lemma 3, we have $|\lambda'(t)|^2\leqq\sup_{|t-t_0|\leqq 2\varepsilon}|B''(t)|/2$ if $|t-t_0|\leqq\varepsilon$. Since $B''(t_0)=0$, we have $\lambda'(t)\to 0$ $(t\to t_0)$.

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