The spectrum of 1-Rotational Steiner Triple Systems over a Dicyclic Group

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Abstract

The spectrum of values v for which a 1-rotational Steiner triple system of order v exists over a dicyclic group is determined.

Key words: Dicyclic group, extended Skolem sequence, 1-rotational difference family, 1-rotational Steiner triple system
PACS: 07.05.Fb

1 Introduction

Let V be a set of v points and $\mathcal{B}$ be a collection of 3-subsets, called *blocks* (or *triples*), of V. A pair $(V, \mathcal{B})$ is called a *Steiner triple system* of order v, denoted by STS(v), if every pair of points is contained in exactly one block.

An *automorphism group* of a Steiner triple system, $(V, \mathcal{B})$, is a group of bijections on V preserving $\mathcal{B}$. An STS(v), $(V, \mathcal{B})$, is said to be 1-*rotational* over a group G if it admits an automorphism group fixing a single point (usually denoted by ∞) and acting regularly on the remaining $v - 1$ points under the action of G. In this case, V is identified with $\{\infty\} \cup G$.

Phelps and Rosa [4] first introduced the concept of a 1-rotational STS(v) and gave the spectrum of 1-rotational STS(v), i.e., the set of values v for which a 1-rotational STS(v) exists, over a cyclic group. Buratti [3] investigated the spectra $\mathcal{A}_{1\mathrm{r}}$, $\mathcal{Q}_{1\mathrm{r}}$ and $\mathcal{G}_{1\mathrm{r}}$ of 1-rotational STS(v) over an abelian group, a

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[1] This work was supported in part by Monbu Kagakusho Grant-in-Aid for Young Scientists (B)17700278 and by JSPS Scientific Research (C)19500236.

dicyclic group and an arbitrary group, respectively. In [3], besides completely determining $\mathcal{A}_{1\mathrm{r}}$, he gave partial answers about $\mathcal{Q}_{1\mathrm{r}}$ and $\mathcal{G}_{1\mathrm{r}}$.

In this paper, the spectrum $\mathcal{Q}_{1\mathrm{r}}$ of 1-rotational STS(v) over a dicyclic group will be completely settled.

Proposition 1 ([3]) *A necessary condition for the existence of a* 1*-rotational STS*(v) *over a dicyclic group is that* $v \equiv 9 \pmod{24}$. *That is,* $\mathcal{Q}_{1\mathrm{r}} \subseteq 24N+9$, *where* N *is the set of nonnegative integers.*

In Section 9 of [3], Buratti conjectured that $\mathcal{Q}_{1\mathrm{r}} = 24N + 9$ and proved

$$(96N + 9) \cup (96N + 33) \subseteq \mathcal{Q}_{1\mathrm{r}},$$

which assures "half" of the sufficiency of Proposition 1. In the next section, the remaining half will be proved. That is,

$$(96N + 57) \cup (96N + 81) \subseteq \mathcal{Q}_{1\mathrm{r}} \tag{1}$$

will be shown.

2 Completion of the Spectrum $\mathcal{Q}_{1\mathrm{r}}$

The *dicyclic group* (also called the *generalized quaternion group*) of order $4t$, denoted by Q_{4t}, has the following defining relations (see [5]):

$$Q_{4t} = \langle x, y \mid x^{2t} = 1, y^2 = x^t, yx = x^{-1}y\rangle.$$

Equivalently, we have

$$Q_{4t} = \{1, x, x^2, \ldots, x^{2t-1}, y, xy, x^2y, \ldots, x^{2t-1}y\}$$

with $x^{2t} = 1$, $y^2 = x^t$ and $yx^i = x^{-i}y$ for any i.

In order to prove $\mathcal{Q}_{1\mathrm{r}} = 24N+9$, we need extended Skolem sequences. Among several ways to describe the definition of an extended Skolem sequence, here we adopt the one in [3].

Definition 2 *Let* k *and* n *be integers with* $1 \leq k \leq 2n+1$. *A* k-extended Skolem sequence *of order* n, *denoted by* k*-ext* $\mathcal{S}_n$, *is a sequence* $(a_1, a_2, \ldots, a_n)$ *of* n *integers such that*

$$\bigcup_{i=1}^{n} \{a_i, a_i - i\} = \{1, 2, \ldots, 2n+1\} \setminus \{k\}.$$

When $k = 2n+1$, *it is simply called a* Skolem sequence *of order* n.

The existence of k-ext $\mathcal{S}_n$ is known for arbitrary k due to Baker [2].

Theorem 3 ([2]) *There exists a k-ext $\mathcal{S}_n$, $1 \le k \le 2n+1$, if and only if either*

(i) *k is odd and $n \equiv 0$ or $1 \pmod 4$, or*

(ii) *k is even and $n \equiv 2$ or $3 \pmod 4$.*

Now we are going to confirm (1).

Theorem 4 *There exists a 1-rotational* STS$(24m+9)$ *over Q_{24m+8} for any $m \equiv 2$ or $3 \pmod 4$. That is, $(96N+57) \cup (96N+81) \subseteq \mathcal{Q}_{1\mathrm{r}}$.*

Proof. Applying Theorem 3, it is trivial to see that for every $m \equiv 2$ or $3 \pmod 4$ there exist a $2m$-ext $\mathcal{S}_m$ and a $3m$-ext $\mathcal{S}_{3m}$.

Now let $(a_1, a_2, \ldots, a_m)$ and $(b_1, b_2, \ldots, b_{3m})$ be the $2m$-ext $\mathcal{S}_m$ and the $3m$-ext $\mathcal{S}_{3m}$, respectively, and take a set of triples as follows:

$$\begin{aligned}\mathcal{F} =&\{\{\infty, 1, x^{6m+2}\}\} \cup \{\{1, x^{a_i-i+m}, x^{a_i+m}\} \mid i = 1, 2, \ldots, m\} \cup \\ &\{\{1, x^{3m+1+j}, x^{b_{3m+1-j}}y\} \mid j = 1, 2, \ldots, 3m\} \cup \{\{1, x^{3m}, x^{6m+2}y\}\}.\end{aligned}$$

By checking the differences arising from $\mathcal{F}$, it can be readily verified that $\mathcal{F}$ is a 1-rotational $(24m+9, 3, 1)$ difference family over Q_{24m+8} and hence it generates a 1-rotational STS$(24m+9)$ (see [1] for the definition and existence results of 1-rotational difference families).

Note that $\mathcal{F} \backslash \{\{\infty, 1, x^{6m+2}\}\}$ is particularly called a 1-rotational $(Q_{24m+8}, \{1, x^{6m+2}\}, 3, 1)$ difference family (for the precise definition, see [3]). $\square$

Remark 5 Since $x^i y(x^{i+t}y) = x^i x^{-(i+t)} y^2 = x^{-t} x^t = 1$,

$$(x^i y)^{-1} = x^{i+t} y$$

holds over Q_{4t}. For instance, over Q_{24m+8}, the differences arising from $\{1, x^{3m}, x^{6m+2}y\}$ are calculated as follows:

$$\begin{aligned}&\{x^{3m}, x^{-3m}, x^{6m+2}y, (x^{6m+2}y)^{-1}, x^{3m}(x^{6m+2}y)^{-1}, x^{6m+2}yx^{-3m}\} \\ &\qquad = \{x^{3m}, x^{9m+4}, x^{6m+2}y, y, x^{3m}y, x^{9m+2}y\}.\end{aligned}$$

Example 6 A 1-rotational STS(57) over Q_{56}. In this case, $m = 2$. Take

$$(2, 5) \quad \text{and} \quad (2, 5, 12, 11, 13, 10)$$

as the 4-ext $\mathcal{S}_2$ and the 6-ext $\mathcal{S}_6$, respectively. Then the short block orbit is represented by $\{\infty, 1, x^{14}\}$, and full block orbits are represented by

$\{1, x^3, x^4\}$, $\{1, x^5, x^7\}$, $\{1, x^8, x^{10}y\}$, $\{1, x^9, x^{13}y\}$, $\{1, x^{10}, x^{11}y\}$, $\{1, x^{11}, x^{12}y\}$, $\{1, x^{12}, x^5y\}$, $\{1, x^{13}, x^2y\}$ and $\{1, x^6, x^{14}y\}$.

Example 7 A 1-rotational STS(81) over Q_{80}. In this case, $m = 3$. Take

$$(2, 5, 7) \quad \text{and} \quad (2, 7, 6, 18, 15, 17, 19, 16, 13)$$

as the 6-ext $\mathcal{S}_3$ and the 9-ext $\mathcal{S}_9$, respectively. Then the short block orbit is represented by $\{\infty, 1, x^{20}\}$, and full block orbits are represented by $\{1, x^4, x^5\}$, $\{1, x^6, x^8\}$, $\{1, x^7, x^{10}\}$ $\{1, x^{11}, x^{13}y\}$, $\{1, x^{12}, x^{16}y\}$, $\{1, x^{13}, x^{19}y\}$, $\{1, x^{14}, x^{17}y\}$, $\{1, x^{15}, x^{15}y\}$, $\{1, x^{16}, x^{18}y\}$, $\{1, x^{17}, x^6y\}$, $\{1, x^{18}, x^7y\}$, $\{1, x^{19}, x^2y\}$ and $\{1, x^9, x^{20}y\}$.

Bringing together Theorem 9.1 in [3] and Theorem 4, we can establish the following.

Theorem 8 *There exists a* 1*-rotational* STS(v) *over a dicyclic group if and only if* $v \equiv 9 \pmod{24}$*. That is,* $\mathcal{Q}_{1\mathrm{r}} = 24N + 9$.

Acknowledgements

The author would like to thank the referees for their comments in improving the readability of the article. She is also grateful to Professor Marco Buratti and Professor Masakazu Jimbo for communicating on this topic with her.

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